PROOF?!

Screenplay by Daniel Radin

This screenplay is adapted from *Proofs and Refutations: The Logic of Mathematical Discovery** by Imre Lakatos (November 9, 1922 – February 2, 1974). The original work, like my version, is in the form of a play taking place in a classroom. I wrote this adaptation of the first part of Lakatos' work for my community college students. I believe that this version, while far less thorough than Lakatos' original work, is easier for the layperson to read. If you find this chapter intriguing, I highly recommend the original work, which is truly a masterpiece on many levels. I am grateful for the late Mr. Lakatos' wonderful creation, and I want to reiterate that he deserves full credit for all of the ideas in this chapter.

Scene 1: The Proof

T: Good morning class! As you may remember, we spent last class studying polyhedra such as these models on my desk. We were trying to find a rule for relating the numbers of edges, vertices, and faces of polyhedra. As you know, for the polyhedra's simpler two-dimensional cousins, the polygons, there is a simple rule. In polygons, such as triangles, squares, pentagons, etc., the number of edges is equal to the number of vertices, and the number of faces is always one. We discovered last time that the rule for polyhedra seemed to be that the number of vertices plus the number of faces minus the number of edges was always two. Or $V + F - E = 2$. (*T* writes on the board explaining what each letter stands for.) By the way, this is known as Euler's Theorem of Polyhedra. But as of last class, no one had come up with a proof. So, as far as we know, our rule was really only a conjecture. Has anybody found a proof?

* This material is adapted from pp. 6-70, Proofs and Refutations: The Logic of Mathematical Discovery by Imre Lakatos edited by John Worrall and Elie Zahar

<u>Euler's Theorem of Polyhedra</u>

V + F – E = 2

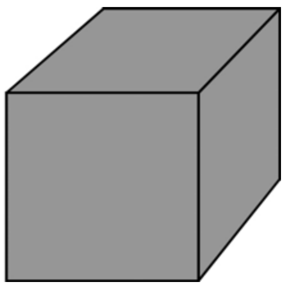
V = number of vertices

F = number of faces

E = number of edges

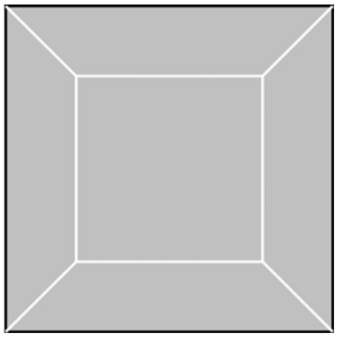
S: I haven't been able to come up with a proof but I'm satisfied that it's true. After all, we must have tried twenty different cases and it always worked. But if you have a proof, I'd like to see it.

T: Actually, I do have a proof, of sorts. It is in the form of a thought experiment. Let's use this rubber cube to demonstrate my proof.

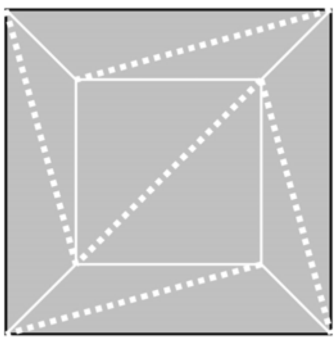


Of course you could use any polyhedron. I will show, in three steps, that the conjecture is true. I am going to prove, without counting, that for this cube, V + F − E = 2. For my first step, I will remove one face. (*T* cuts off one face with a knife.)

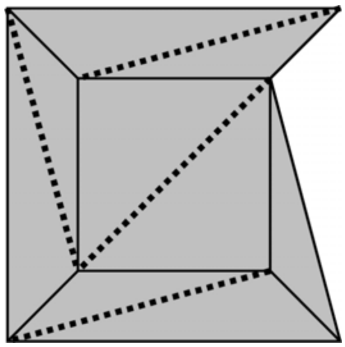
Now, if and only if it was originally true that V + F − E = 2, then now V + F − E = 1, since I have certainly reduced F by one and hence reduced V + F − E by one from two to one. Can you see that V and E haven't changed? Now I staple the resulting defaced polyhedron onto a flat wooden board and mark the edges with white-out. (*T* proceeds to do this with a staple gun and some white-out.)



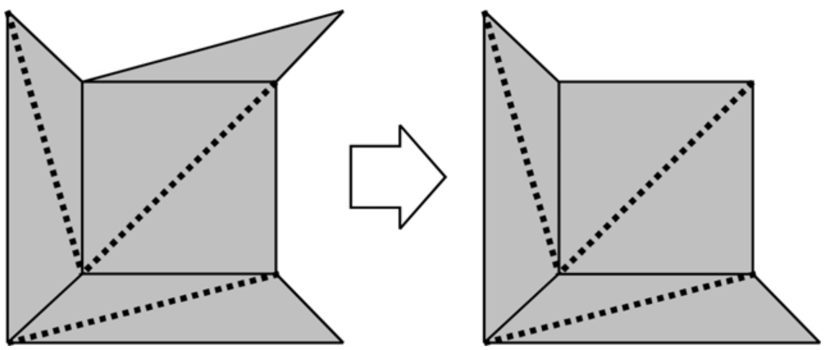
I contend that I could theoretically do this with any polyhedron. The resulting picture would be a map of the original polyhedron minus one face. For my second step, I draw diagonals in each polygonal face until the map is reduced to all triangles.



But you see that every time I add a diagonal, I increase the number of faces by one and I also increase the number of edges by one. These will cancel each other out and keep the expression, V + F − E, unchanged. Before my third step, I must first transfer the resulting map onto the board, since I will need to do some erasing. Now, I will begin removing triangles one by one. There are two ways to remove triangles. The first way involves removing one edge.



Thus we have lost one edge and one triangular face. Or E goes down by one and F goes down by one and you see, V + F − E doesn't change. The second way to remove a face is to remove two edges and a vertex like so.



So E goes down by two, V goes down by one, and of course, F goes down by one, since we are removing a triangular face. Thus V + F − E remains unchanged. If we keep removing triangles, we will eventually end up with just one triangle. Now we know that for any triangle, V + F − E = 1. Since V + F − E remained unchanged throughout the triangle removal process, it must have equaled one ever since that top face was cut off of the cube. So you see, V + F − E had to have equaled two in the original polyhedron. Therefore, I have proven the conjecture that V + F − E = 2 for any polyhedron.

Scene 2: The Objections

D: Now that you've proven it, you don't have to call it a conjecture anymore. Now it's a theorem.

A: I'm not convinced. Are you sure that this works for any polyhedron? For instance, I wonder about your first step. Can any polyhedron be stretched flat on a board after the removal of one face?

B: Yeah, also, in that second step, I'm not convinced that every time you add a diagonal you get a new face.

G: And that third step, are you sure that there are only two ways that triangles can be removed? And are you sure that you will always end up with one triangle?

T: No, I'm not sure of any of these things.

A: So now we're worse off than before. Now we have three conjectures to prove instead of just one. How can you call what you just did a proof?

T: Well, if what you mean by a "proof" is something that establishes the truth of a conjecture, then I guess my thought experiment doesn't fit your definition.

D: Then what do you think a proof does?

T: That's a tricky question which I hope we will get to. For now, I propose to use the word, "proof," to describe a thought experiment that breaks a conjecture down into smaller sub-conjectures known as lemmas. By doing this, we have created a broader front by which to attack the original conjecture. Now we can look for counterexamples for any of the three sub-conjectures as a way of attacking the original conjecture.

<u>Agenda</u>
1. What does a proof do?

Scene 3: Arguing the Third Lemma

G: As I said before, I suspect the third sub-conjecture or lemma as you call it. I suspect that there are other ways to remove triangles.

T: Suspicion is not a valid criticism.

G: What if I have a counterexample?

T: Conjectures ignore suspicion but they cannot ignore a counterexample.

G: Here's a counterexample. What if I remove a triangle from the inside of the network of triangles? Now I have removed a face without removing any edges or vertices. So I have changed V + F − E and hence the third lemma is false! (*G* shows on board.)

T: You're right. But notice that while even a cube can be seen as a counterexample to my third lemma as you have shown, a cube is still not a counterexample to the original conjecture that V + F − E = 2. So you have a valid criticism of the proof, but not of the conjecture.

A: Does this mean that you will give up on this proof?

T: No, I will just improve the proof to stand up to the new criticism.

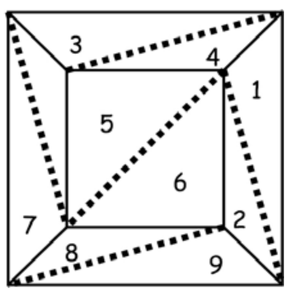
G: How?

T: First, let me explain a few new terms. There are two kinds of counterexamples: local counterexamples and global counterexamples. A local counterexample is an example that contradicts a lemma from within the proof, and a global counterexample is an example that contradicts the main conjecture that you're trying to prove. So you see, your counterexample is local but not global. It is a valid criticism of my proof but not of the original conjecture that $V + F - E = 2$.

G: So the conjecture may still be true, but I have shown that your proof does not prove it.

T: Yes, but I can easily fix my proof, or in particular, the lemma in question, so that your counterexample will no longer refute it. I only need to specify that the triangles must be removed from the boundary of the network, not from the inside. For instance, I could word it something like this: "Now remove the boundary triangles from the network one by one." So you see, it only took one small obvious adjustment to fix my proof.

G: I don't think it was such an obvious adjustment. It was actually pretty clever. Now I will show you that it was also false. If I remove boundary triangles as your new proof tells me I can, I can still run into trouble**.** What if I remove them in this order? (*G* shows at the board.) Now, as I remove this eighth triangle, I am removing two edges, one face, and no vertex, and hence $V + F - E$ increases by one. If a boundary triangle is a triangle along the boundary, then you can't claim that this eighth triangle is not a boundary triangle.



T: I could say that by "boundary" triangle I meant a triangle that does not disconnect the network, but honesty keeps me from doing this. Mathematical arguments starting with "I meant…" are rarely completely honest since part of the idea of a mathematical argument is to be perfectly clear about what you mean whenever you say anything. So yes, you got me. Here is a third version which will stand up to both of your counterexamples: "Remove the triangles one by one in such a way that V + F − E does not change."

K: Yes, your new lemma is certainly true. It says that if we remove triangles in such a way that V + F − E does not change, then V + F − E does not change. Big surprise there!

T: No, the lemma says that there is always an order for removing the triangles one by one without changing V + F − E until you get to the last triangle.

K: But how will we know what order to use and even if such an order exists? You started out with a thought experiment with definite instructions about removing triangles. Then you changed it to boundary triangles. Now you say to follow some particular order. But you don't say what the order is. How can we perform your experiment, even if it is a thought experiment, if you don't tell us exactly what to do? Your new lemma beats the counterexamples. But your thought experiment is no longer valid, and hence you no longer have a proof.

R: Actually, only the third step is gone.

K: You know, I'm not even sure I would call your new lemma an improvement. The first two versions looked true before we found the counterexamples. This last one is just a temporary patch. Do you really think this one will survive?

T: Single statements that look true are often quickly disproved. But more sophisticated statements that have been through several generations of criticism are more likely to actually end up being true.

O: What happens if this new more sophisticated lemma turns out to be false and you can't come up with a new patch?

T: Good question! Let's put it on the agenda for tomorrow.

<u>Agenda</u>

1. What does a proof do?

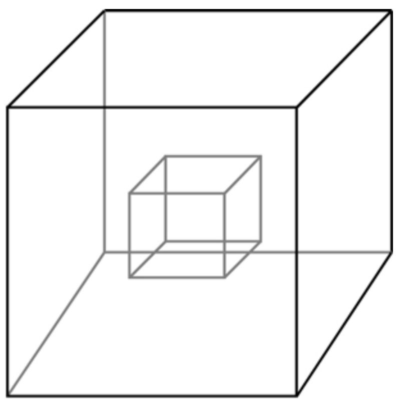
2. What happens if we can't patch the lemma?

Scene 4: Counterexamples

A: I have a counterexample that refutes the first lemma. And it also refutes the main conjecture. That is, it can't be cut open and stretched flat, and it also doesn't satisfy V + F − E = 2. So it is a global counterexample and a local counterexample.

T: Great! Tell us what it is.

A: It is a solid defined as a cube with a cube-shaped hollow inside it like this. (*A* draws it on the board.) You can see that no matter what face you remove, it still won't stretch out into a flat map of edges and vertices. Also since for each cube, V + F − E = 2, my new shape must have V + F − E = 4. So it violates the first lemma and the original conjecture.



counterexample 1

T: Good job! We'll call it counterexample 1. (*T* labels it on the board.) Now what?

G: How can you take this so calmly? *A* has just wiped out your proof and the conjecture you're trying to prove. One counterexample is all it takes. It's time to scrap the whole conjecture.

T: I admit that the conjecture has taken a severe hit. I still believe that my proof, however, was successful. Remember that when I speak of a proof, I mean a way of breaking down a conjecture into a thought experiment containing several smaller conjectures called lemmas. These lemmas make it easier to analyze and challenge the original conjecture. Here, I think my proof was very successful. My proof certainly helped us learn more about the original conjecture.

A: So a local counterexample is a criticism of the proof, and does not hurt the conjecture, and a global counterexample hurts the conjecture, but does not invalidate the proof. But if a global counterexample knocks out the conjecture, what is left for the proof to prove?

G: Yeah, if the conjecture is gone, everything must go, including the proof!

D: But why do we have to accept this counterexample? The conjecture has been proven. It is now a theorem. It may not describe this so-called counterexample, but why should it give way? Let's get rid of the counterexample instead. I say this pair of nested cubes isn't a polyhedron at all. It's a monster created by *A*, and therefore it does not contradict the theorem.

A: Sure it's a polyhedron. A polyhedron is a solid bounded by polygonal faces.

T: Let's call this Definition 1. (*T* writes it on the board.)

<u>Polyhedron</u>
Definition 1: a solid bounded by polygonal faces.

D: Your definition is wrong. A polyhedron is a surface, not a solid. It has faces, edges, and vertices. And it can be stretched out on a board. The proper definition of a polyhedron is a surface consisting of a system of polygons.

T: Call it Definition 2. (*T* writes it on the board.)

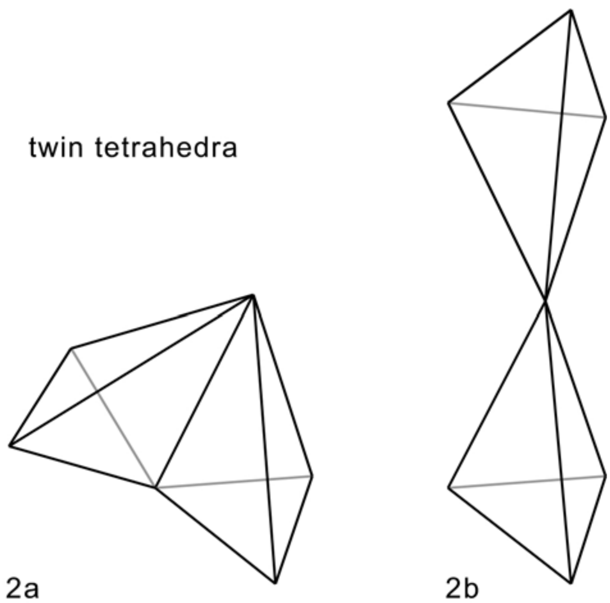
<u>Polyhedron</u>
Definition 1: a solid bounded by polygonal faces.
Definition 2: a surface consisting of a system of polygons.

D: So your so-called counterexample was really two polyhedra, one inside the other, i.e., two cubes. That would be like saying that a woman pregnant with a baby inside her is a counterexample to the conjecture that people have only one head each.

A: So my counterexample has made you come up with a new definition for polyhedron. Or is it your claim that you always meant a surface by saying polyhedron?

T: Let's just accept *D*'s new definition as Definition 2. Now that we mean a surface when we say polyhedron, can you still come up with a counterexample to refute the conjecture?

A: Actually, I can think of two. Take two tetrahedra that have a vertex in common or two tetrahedra that have an edge in common. (*A* draws them.) In each case, V + F – E = 3.



T: We'll call them counterexamples 2a and 2b. (*T* labels them.)

D: Very imaginative! But of course, I didn't mean any system of polygons. I meant a system of polygons arranged so that exactly two polygons meet at any edge and there is a route inside the polyhedron from the inside surface of any polygon to the inside surface of any other polygon without crossing any edges or vertices. You can see how this knocks out your two sets of Siamese twin tetrahedra. (*D* demonstrates on the board.)

T: Call that Definition 3. (*T* writes it on the board.)

Polyhedron

Definition 1: a solid bounded by polygonal faces.

Definition 2: a surface consisting of a system of polygons.

Definition 3: a surface consisting of a system of polygons arranged so that exactly two polygons meet at any edge and there is a route inside the polyhedron from the inside surface of any polygon to the inside surface of any other polygon without crossing any edges or vertices.

A: You are the imaginative one, making up one definition after another to protect your pet theorem from my counterexamples. Why don't you just define a polyhedron as a system of polygons for which $V + F - E = 2$? This definition would settle the dispute forever!

T: Call that Definition P. (*T* writes it on the board.)

<u>Polyhedron</u>

Definition 1: **a solid bounded by polygonal faces.**

Definition 2: **a surface consisting of a system of polygons.**

Definition 3: **a surface consisting of a system of polygons arranged so that exactly two polygons meet at any edge and there is a route inside the polyhedron from the inside surface of any polygon to the inside surface of any other polygon without crossing any edges or vertices.**

Definition P: **a surface consisting of a system of polygons for which $V + F - E = 2$.**

A: Then there would be no reason to study the subject any further.

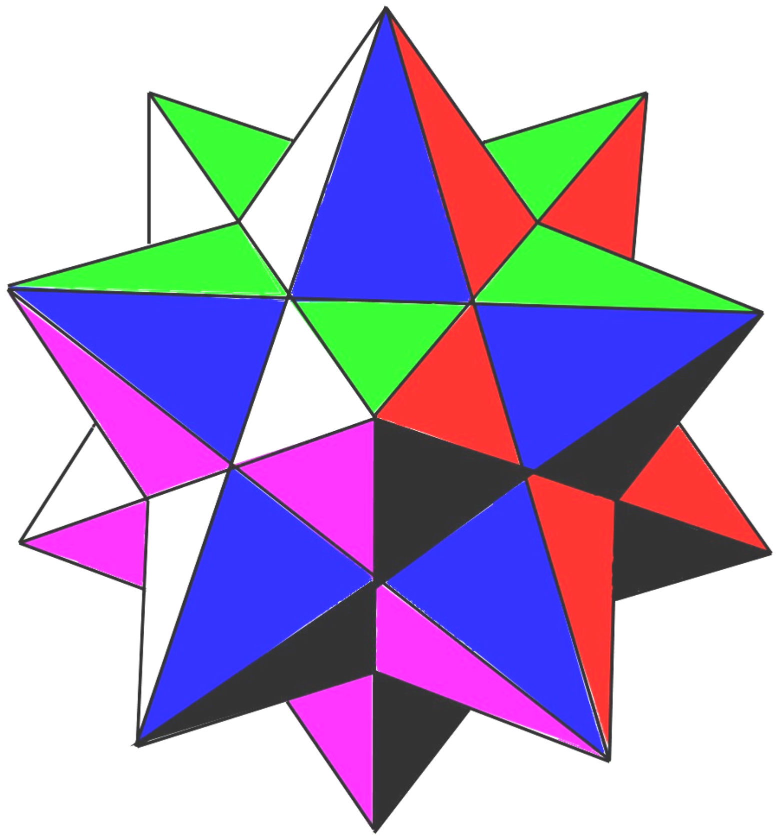
D: I get your point but, on the other hand, any theorem can be contradicted by cleverly constructed monsters such as yours.

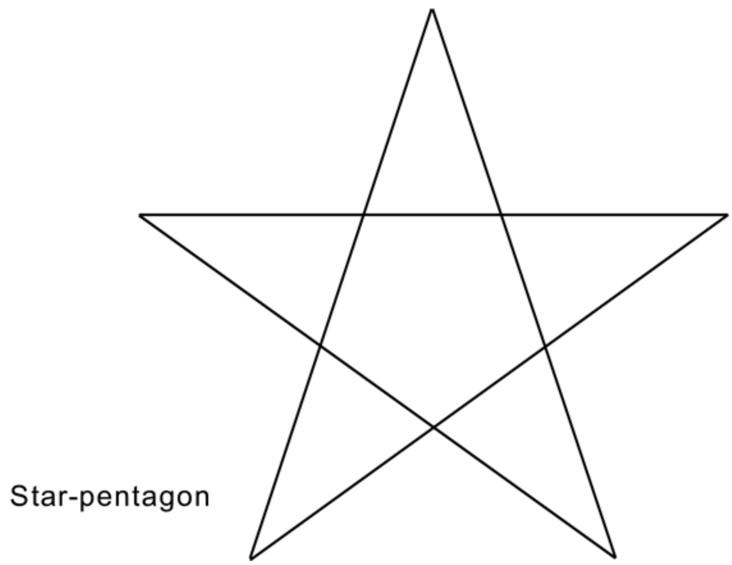
T: As you can see, when you come up with counterexamples to a conjecture, you often end up arguing about the definitions of the terms in the conjecture. In our conjecture, the ambiguity seems to be in the definition of the term, polyhedron. I made the mistake of assuming that we all agreed on what is and what is not a polyhedron. For now, let's not argue about which is the proper definition. Let's assume all of the definitions together. Does anybody have a counterexample that would work for even the most restrictive definition of a polyhedron?

K: Including Definition P?

T: No, not including Definition P.

G: I have one. I call it "the urchin." It's a star-polyhedron. Here's a model of it. (*G* displays model.) It consists of twelve star-pentagons like this one. (*G* shows model of a star-pentagon.) It has 12 vertices, 12 star-pentagonal faces, and 30 edges. You can count and see. So therefore V + F − E equals 12 + 12 − 30 which is −6, not 2. (*G* writes all of this on the board.)





12 Vertices 12 Faces 30 Edges

V + F – E = 12 + 12 – 30 = –6 not 2

D: What makes you think your “urchin” is a polyhedron?

G: Don’t you see? It’s a polyhedron whose faces are star-polygons. So it consists of a system of polygons as required by definitions one and two. Exactly two polygons meet at any edge and it is possible to get from the inside of any polygon to the inside of any other polygon without crossing any edges or vertices. So you see, it satisfies Definition 3.

D: Then I guess you don’t even know what a polygon is! A star-polygon is not a polygon. A polygon is a system of edges arranged so that exactly two edges meet at any vertex and the edges have no points in common except the vertices.

T: We’ll call that Definition 1. (*T* writes it on the board.)

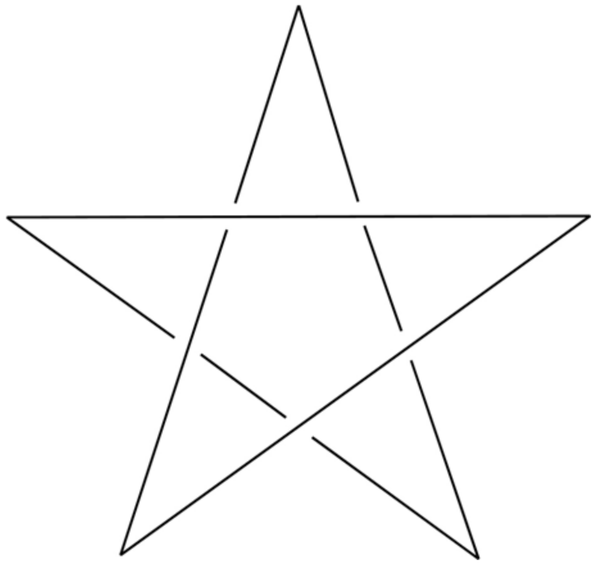
<u>Polygon</u>	
Definition 1:	**a system of edges arranged so that exactly two edges meet at any vertex and the edges have no points in common except the vertices.**

G: I agree that exactly two edges meet at any vertex. But I don't see why the edges have to have no points in common besides the vertices. I think the correct definition is just the first half of your definition.

T: We'll call that Definition 1'. (*T* writes it on the board.)

<u>Polygon</u>	
Definition 1:	**a system of edges arranged so that exactly two edges meet at any vertex and the edges have no points in common except the vertices.**
Definition 1':	**a system of edges arranged so that exactly two edges meet at any vertex.**

G: Look, if I just lift the edge of this model of a star-pentagon, it still satisfies the entire Definition 1 anyway. Now the edges have no points in common except the vertices. Your problem is that you limit yourself to the plane. You should let your polygons stretch out into space.

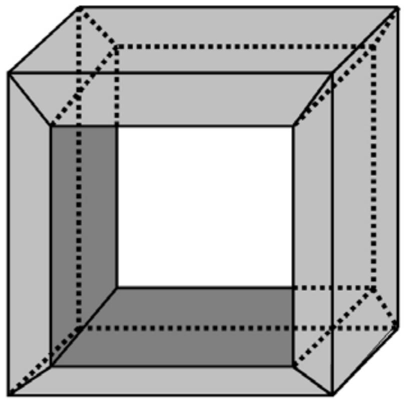


D: Can you tell me what is the area of your star-polygon, or do you claim that polygons don't have areas?

G: You were the one who claimed that polyhedra were not solids, that they were just surfaces. In that case polygons must be just closed curves consisting of edges and vertices, and not the area they enclose.

T: Let's save this debate for another time and get back to the task at hand. Does anyone have a counterexample that works for even the new definitions: 1 and 1'?

A: I have one. Look at this picture frame. (*A* shows the frame.) It passes every definition of a polyhedron. But if you count its edges, faces, and vertices, you will see that you get $V + F - E = 0$.



counterexample 4 (picture frame)

T: We'll label it counterexample 4. (*T* puts a label on it.)

B: I guess that does it. We've just been wasting our time. The conjecture is false.

A: Well *D*, aren't you going to say anything? Can't you define this new counterexample out of existence? Have you given up? Do you admit that we have finally shown the existence of non-Eulerian polyhedra?

Scene 5: Order versus Disorder

D: *A*, I am getting tired of your little non-Eulerian pests. You should really find another name for them though, instead of insisting on calling them polyhedra. I see Euler's theorem as a beautiful example of the order and harmony in mathematics. You seem to prefer to seek anarchy and chaos in mathematics. I don't see how we can ever resolve our differences.

A: Well you will say anything to preserve your precious order and harmony from supposed anarchists like myself.

T: Do we have a new definition to rescue the conjecture?

A: You mean the latest contraction of the concept of polyhedron. *D* simply sidesteps real problems with new definitions. He doesn't solve them.

D: I'm not contracting the concept. You're expanding it. For instance, your picture frame is obviously not a polyhedron.

A: Why not?

D: If you cut your picture frame with a plane like so, (*D* cuts frame on table saw.) you see that it has two completely disconnected polygon cross-sections. You will find that this is true for any plane passing through the inside of the frame.

A: Your point being…?

D: For a genuine polyhedron, there is at least one plane through any point such that the intersection of the plane with the polyhedron consists of a single polygon.

T: We'll call that Definition 4. (*T* writes it on board.)

<u>Polyhedron</u>

Definition 1: **a solid bounded by polygonal faces.**

Definition 2: **a surface consisting of a system of polygons.**

Definition 3: **a surface consisting of a system of polygons arranged so that exactly two polygons meet at any edge and there is a route inside the polyhedron from the inside surface of any polygon to the inside surface of any other polygon without crossing any edges or vertices.**

Definition P: **a surface consisting of a system of polygons for which V + F − E = 2.**

Definition 4: **a surface consisting of a system of polygons arranged so that exactly two polygons meet at any edge and there is a route inside the polyhedron from the inside surface of any polygon to the inside surface of any other polygon without crossing any edges or vertices and there is at least one plane through any point such that the intersection of the plane with the polyhedron consists of a single polygon.**

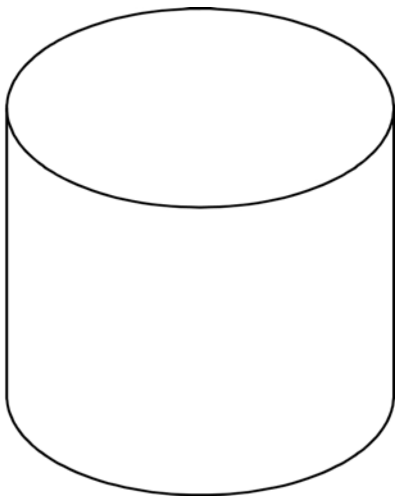
A: For each counterexample, you have a new definition which you claim to be just a deeper insight into the original concept of polyhedron. You have turned Euler's original beautiful conjecture that V + F − E = 2 into some sort of holy dogma to be followed blindly. (*A* leaves the room.)

D: I can’t understand why an intelligent mathematician like *A* wastes her(his) time with these monsters (s)he calls polyhedra. Monstrosities never serve any purpose either in nature or in the world of thought. Nature always follows a harmonious orderly pattern.

G: Biologists would argue with that. The theory of evolution is driven by mutations sometimes referred to as “hopeful monsters.” I think *A*’s counterexamples are “hopeful monsters.”

D: Well, *A* is gone and there will be no more monsters of any sort.

G: I’ve got a counterexample. It satisfies every definition we have, and yet $V + F - E = 1$. My counterexample is a cylinder. (*G* provides cylinder.) It has three faces: the top, the bottom, and the side, two edges: the circles, and no vertices.



D: *A* stretched concepts, but you tear them. An edge has to have two vertices.

T: Is this another definition?

G: Why exclude edges with one or even zero vertices? You used to contract concepts. Now you practically define them out of existence.

D: I don't see any purpose for your so called counterexamples. You're trying to include so many weird monsters into the term, "polyhedron," that there is barely any room for ordinary named polyhedra.

G: I believe that the only way to gain a deep understanding into a concept is to study it at the edges where things get more real and interesting. If we want to know about what a polyhedron is, we must look at the edge, the lunatic fringe, of polyhedra.

T: I must agree that while *D* has done an excellent job of defending the theorem against monstrous counterexamples, his method is perhaps not the most useful. We need to somehow treat these monsters with more respect. *D* is still holding to the idea that a proof must prove what it sets out to prove. I see a proof as the decomposition of a conjecture into subconjectures. Hence, even if the conjecture is false, the proof can still be interesting.

Scene 6: Exceptions

B: That was very confusing. But before you explain it further, there is something I have to say.

T: Yes, let's hear it.

(*A* re-enters.)

B: I don't think we should call these things monsters or even counterexamples. This just sets them up as the enemy. I think they are natural and I propose that we call them exceptions. They help to restrict the domain of the conjecture.

S: I agree with you. I think there are three kinds of conjectures: those that are always true, those that are false, and those that are generally true with certain exceptions.

E: What are you talking about?

S: There is a difference between a false conjecture and one that is subject to restrictions. After all, they say “the exception proves the rule.”

E: (to *K*) (S)he needs to learn something about logic.

D: I am embarrassed to say that I think *A* and I are on the same side in this discussion. At least we both agree that the conjecture must be either true or false. *S*’s idea of a third category: true but subject to exceptions, makes no sense. You can’t have a mathematical theorem that has exceptions. It’s too muddy for mathematics. It’s starting to sound like English grammar.

A: I agree.

E: I must side with *D*’s original arguments against the monsters. It’s not so much a matter of protecting the theorem from the likes of *A*. My interest is in protecting mathematics from *S*’s kind. I don’t think we need exceptions if we are careful with our definitions.

A: You could just as easily side with my counterexamples and label the conjecture false.

E: It makes more sense to me to reject your monsters than to reject a perfectly good proof.

T: Let’s go back to *B*’s and *S*’s idea of renaming the counterexamples as exceptions.

Scene 7: Refining the Domain

B: Actually, I don’t agree with *S*’s idea of a third category vaguely named, “true with exceptions.” I see the exceptions as useful tools, not for attacking the conjecture, but for refining it and narrowing it down until we know exactly where it does and does not apply.

T: So what would be the precise domain of Eulerean polyhedra?

B: All polyhedra that have no cavities (like the nested cubes) and no tunnels (like the picture frame).

T: Are you sure?

B: Yes, I'm sure.

T: What about the twin tetrahedron? (*T* points them out.)

B: Okay, no cavities, no tunnels, and no multiple structures.

T: I like your idea, but I wonder if it is really now perfect and unambiguous. How do you know it excludes all exceptions?

B: Can you name one I don't exclude?

A: What about my urchin? (*A* points to it.)

G: What about my cylinder? (*G* points to it.)

T: Even if we don't have an actual exception to show you, how can you be sure none exist?

B: I guess you're right. It was ridiculous to think we could generalize from our small study of regular polyhedra. I'm surprised we didn't find more exceptions. I think we can safely restrict our domain to convex polyhedra though. It was the concave ones that gave us trouble.

G: What about my cylinder? It's convex.

B: It's a joke.

T: Setting aside the question of the cylinder, I still have some criticism of your method. You have retreated for safety to include just convex polyhedra. But perhaps you have gone too far and excluded many fine Eulerean polyhedra. The original conjecture may have been an overstatement. But yours sounds like an understatement. At the same time, how do you know that there are no convex exceptions? Perhaps yours is also an overstatement. My other problem is that yours sounds like a guess. Where is the proof? Are you saying we no longer need a proof?

B: I didn't say that.

T: That's true, but you did discover that the proof didn't prove the original conjecture. Does it prove your new conjecture?

B: Well…

E: Here we see that we must not abandon the theorem just because of a few monsters disguised as exceptions.

B: Actually, I reject the original conjecture and its supposed proof. Both have exceptions. But I will restrict both the conjecture and the proof to a proper domain, thereby creating a true and rigorous theorem and proof. For instance, not all polyhedra can be stretched flat on a plane after having one face removed, but all convex polyhedra can be. So the proper theorem is that all convex polyhedra are Eulerian. And it can be rigorously shown that each lemma of the proof holds up under the restricted domain.

T: How do you know that this convex polyhedron restriction isn't just guesswork like the tunnel and cavity guesses?

B: This time it is insight, not guesswork!

T: I admire a guess because it shows courage and modesty. Insight, I question.

B: Question all you like, but do you have a counterexample to my theorem that all convex polyhedra are Eulerian?

T: Certainly, you have no way of being sure that I don't. There is no proof in your method.

B: Do you have the perfect method?

T: No, but I think I can at least show you a method that incorporates counterexamples and proof.

B: I'm all ears.

R: May I get a word in here?

T: Go ahead.

Scene 8: Refining the Interpretations

R: I reject *D*'s method of disqualifying supposed monsters. I also reject *B*'s method of calling them exceptions. I believe that if we look at the supposed monsters and exceptions closely, we will find that they do in fact satisfy Euler's theorem.

T: Really…

A: What about my urchin (*A* points it out.) with its 12 star-pentagon faces?

R: I don't see 12 star-pentagons. I see 60 ordinary triangular faces, 90 edges, and 32 vertices. Hence $V + F - E = 2$ just as Euler predicted. There are no monsters, just monstrous interpretations. You just need to correctly recognize and interpret what you are seeing.

A: I've heard enough of this brainwashing. *T*, please show us your method.

T: Let *R* go on.

R: I have made my point.

O: I don't understand. Surely our goal is to find out exactly which polyhedra satisfy the condition that $V + F - E = 2$.

Scene 9: Generalizing the Problem

Z: No, our problem was just to find out what relationships exist between V, E, and F for polyhedra in general. The fact that we happened to have stumbled on the Eulerian polyhedra first should not limit our study. We saw soon enough that there are at least as many non-Eulerian polyhedra as there are Eulerian. Why not look at when V + F − E = −6 or 28 or even 0? Why are they any less interesting?

S: You're right. We studied V + F − E = 2 because we thought it was true. Now that we know it isn't, we must look for a deeper more basic conjecture, one that will be true for all polyhedra.

O: Let's first solve the Euler problem before we move on. I want to understand exactly why some polyhedra are Eulerian before we look at more general questions. I want to find the secret of Eulerianess!

Z: I understand your resistance, *O*. You have fallen in love with the problem of finding out where God drew the boundary between Eulerian and non-Eulerian polyhedra. But how do you know there is such a line? Maybe Eulerianess is just an accidental property of some polyhedra with no great mystical ramifications. Perhaps Eulerianess is not part of some great order in the Universe.

S: Now we're really lost. With Eulerianess gone, what chance can we possibly have of finding any new order in the chaos of polyhedra and the relationships between vertices, edges, and faces?

B: We found the Eulerian pattern. Surely if we make an organized list of all the polyhedra we have found, we can find a new pattern and then work from there.

Scene 10: Induction versus Deduction

Z: Is that how you think mathematics is created? Just trying one guess after another hoping to stumble upon a pattern?

B: Yes, mathematical knowledge always starts with observation and some insightful discovery. Deductive reasoning only starts after the initial inductive phase.

S: Our first discovery of V + F − E = 2 was sheer luck. And still, it ended in a mess. It is even less likely that we'll come upon anything useful a second time.

B: How else can we start?

Z: I don't need any data to start. I have neither the time, money, nor interest to catalogue and categorize every last polyhedron and then test one formula after another.

B: What will you do then? Lie down on a couch, shut your eyes, and wait?

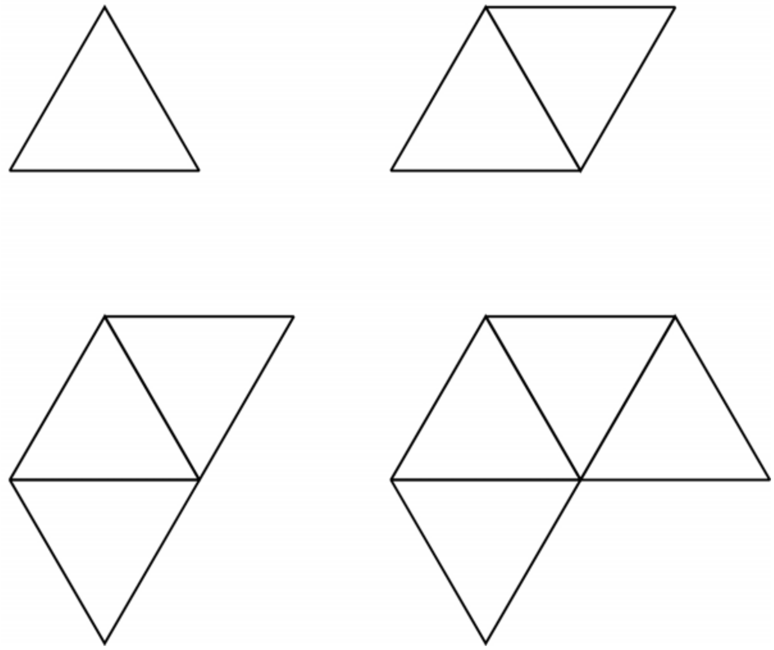
Z: Exactly, I must start with an idea.

B: And where will this idea come from?

Scene 11: First Principles

Z: The idea is already in our minds. It comes from the background knowledge that we already possess. In this case, we knew that for any polygon, V = E. A polygon is a system of polygons containing only one polygon. A polyhedron is a system of polygons containing more than one polygon. For a polyhedron, V does not equal E. So we need to look at why, when we go from one to more than one polygon, V suddenly stops equaling E.

S: So we start with E = V or E − V = 0. Adding a polygon, two edges become one and four vertices become two. So E goes down by one and V goes down by two causing E − V to go up by one. Hence now E − V = 1. (*S* demonstrates with model.) No matter how they go together, we will always loose one more vertex than edges, so E − V will always go up by 1. If E − V goes up by one, V − E goes down by one. Of course F will always go up by one. So V + F − E will stay the same. Thus V + F − E = 1 for a single polygon and it will remain one as we add faces.



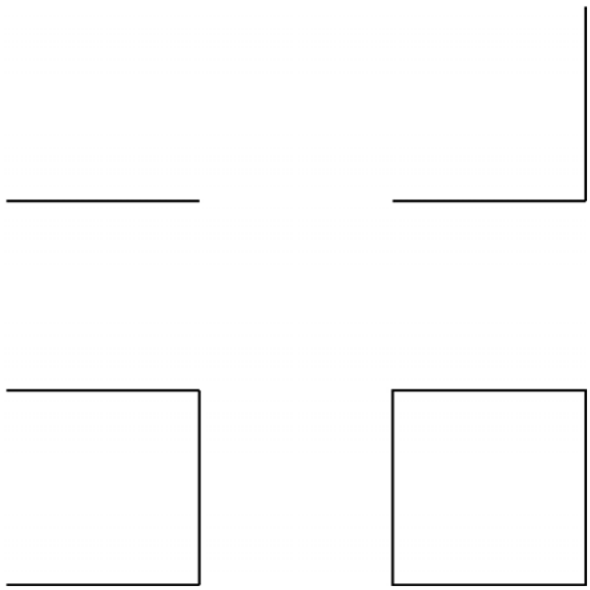
C: But polyhedra have V + F – E = 2.

S: Yes, but that is because my system only leads to open polyhedra with one open face. Once the polyhedron is closed with the final capping face, F increases by one making V + F – E = 2.

Z: So you see, I did not need to start with inductive reasoning.

B: I disagree. You merely pushed back the observation. Your starting point, that V = E for polygons, was certainly an inductive start. How, in fact, did you get V = E?

Z: I was deeply shocked when I realized that V – E = 0 for the triangle. I knew, of course, that for one edge, V – E = 1. I also knew that adding an edge increases both V and E by one. (*Z* shows with model.) Thus V – E remains equal to one. Then I realized that this method will only result in open systems of edges. When I close the system to make a polygon, I add one edge but no vertices and hence V – E decreases from one to zero.



B: Now you've just pushed it back further. Didn't you inductively observe that $V - E = 1$ for an edge?

Z: Actually I started with $V = 1$ for a point.

B: So wasn't that an inductive beginning?

Z: I suppose if I said I started with empty space, you would accuse me of observing nothing.

T: We need to close this discussion for now.

S: We can't stop here. Nothing has been settled.

T: Mathematical inquiry begins and ends with questions.

B: But I didn't start with questions. Now I have nothing but questions.